Pergamon

Computers Math. Applic. Vol. 35, No. 8, pp. 1–7, 1998

0898-1221/98 \$19.00 + 0.00

PII: S0898-1221(98)00040-6

The Stability and Design of Nonlinear Neural Networks

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(Received January 1997; revised and accepted January 1998)

Abstract—Based on the techniques of singular value decomposition and generalized inverse, two new methods for designing associative memories are presented. The two methods not only guarantee that each given vector is an equilibrium point of the network, but also guarantee the asymptotic stability of the equilibrium points. Examples show the effectiveness of the new methods.

Keywords—Neural networks, Asymptotic stability, Associative memory.

1. INTRODUCTION

In recent years, the Hopfield neural network has attracted many researchers, and it has been used in many fields, such as intelligent control, optimization of combinatorial kind, pattern recognition, and associative memories (AM), etc. AM is designed to store a set of vectors, say $\vec{x}$, in such a way that a stimulus, say $\vec{y} = \vec{x} + d\vec{x}$, evokes the output $\vec{x}$ for sufficiently small $d\vec{x}$. If $d\vec{x}$ is considered to constitute either noise or perturbations, then the AM is performing the functions of noise suppression or error correction, respectively, [1].

The Hopfield neural network is a candidate for information processing systems because its dynamical behavior exhibits stable states, which act as basins of attraction towards which neighboring states develop in time. Therefore, the neural network has the ability to retrieve a pattern stored in memory in response to the presentation of an incomplete or noisy version of that pattern. It is error-correcting in the sense that it can override inconstant information in the cues presented to it. Thus, the Hopfield network may be viewed as an associative memory. It is known that an efficient associative memory should store a large set of patterns as memories. During the period of recall, the memory is excited with a key pattern containing a portion of information about a particular member of a stored pattern set. At least, associative memories should have the following properties.

(i) Each desired memory should be stored in the designed network.
(ii) Each desired stored memory must be an asymptotically stable equilibrium of the designed network.

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There are also other properties of effective associative memories, for example, the designed neural network should be of higher capacity, extraneous memories should be minimized, etc. In this paper, the synthesis procedure we proposed shall satisfy (i),(ii).

There have been many designing methods for associative neural networks [1–3]. In [4,5], some of the synthesis procedures proposed for AM are summarized. Unfortunately, the project learning rule based on the techniques of generalized inverse can only guarantee that (i) holds, but can not guarantee that (ii) holds. In this paper, a new rule is proposed. Our synthesis procedure not only guarantee that (i) holds, but also guarantee that (ii) holds. Therefore, it guarantees the attraction of each stored memory. Also, most of the existing methods have serious drawbacks: the interconnected matrix must be a symmetric matrix. In practice, it is almost impossible for us to implement the symmetric architecture in hardware [5].

Using the new method, the interconnected matrix we get can be nonsymmetric. Therefore, it is easier for us to implement it in hardware. Using the technique of singular value decomposition (SVD), a special class of neural network was analyzed in [2], and a synthesis procedure is proposed. In this paper, we first extend the results in [2] via the method of SVD. Then, we give a sufficient condition with the method of generalized inverse in matrix theory. We also present some examples to show the effectiveness and the restriction of our results.

LEMMA 1.1. *(See [6,7].) Assume that $P \in C^{m\times n}$, $Q \in C^{p\times q}$, $D \in C^{m\times q}$, the equation $PXQ = D$ has a generalized solution if and only if there exists $P^{[1]}$, $Q^{[1]}$ such that $PP^{[1]}DQ^{[1]}Q = D$. The general solution of the equation is*

$$X = P^{[1]}DQ^{[1]} + Y - P^{[1]}PYQQ^{[1]},$$

where $Y \in C^{n\times p}$ is an arbitrary matrix, and $P^{[1]}$, $Q^{[1]}$ are the generalized inverses of P, Q, respectively, i.e., they satisfy $PP^{[1]}P = P$, $QQ^{[1]}Q = Q$.

2. DESIGN OF CONTINUOUS-TIME NEURAL NETWORKS

In this paper, we consider the following continuous-time Hopfield neural network:

$$\dot{x} = -Ax + Tf(x) + I, \tag{1}$$

where $x \in R^n$, x_i $(i = 1, 2, \dots, n)$ is the state variable of neuron i; $f(x) = (f_1(x_1), f_2(x_2), \dots, f_n(x_n))^\top$, and $f_i(x_i)$ is differentiable with $f_i'(x_i) > 0$, for $\forall x_i \in R$ and $i = 1, 2 \dots, n$; $I = (I_1, I_2, \dots, I_n)^\top$ is the input vector of the neural network; A, T are $n \times n$ matrices and A is a diagonal matrix with its diagonal element $a_{ii} \geq 0$.

Next, we shall use the techniques of singular value decomposition (SVD) and the generalized inverse (GI) to investigate the design problem for the continuous-time Hopfield neural network.

Suppose we store vectors $x_i^s \in R^n$, $i = 1, 2, \dots, m$, in the network as equilibrium points. Placing these vectors as columns of a matrix, we obtain a matrix $B = [x_1^s, x_2^s, \dots, x_m^s] \in R^{n\times m}$, then T must satisfy the following equation:

$$Tf(B) = AB - I_{n\times m}, \tag{2}$$

where $I_{n\times m}$ is a matrix with vector I as its columns, and $f(B) := [f(x_1^s), f(x_2^s), \dots, f(x_m^s)]$. In order to find matrix T, we apply singular value decomposition (SVD) to $f(B)$.

$$f(B) = U\Sigma V^\top, \tag{3}$$

where U is an $n \times n$ unitary matrix, V is an $m \times m$ unitary matrix (i.e., $U^\top = U^{-1}$, $V^\top = V^{-1}$), and Σ is an $n \times m$ block-diagonal matrix containing the singular values of $f(B)$ (i.e.,

$\Sigma = [\sigma_{ij}]_{n\times m}$, σ_{ij} are equal to zero except $\sigma_{ii} > 0$, $i = 1, 2, \ldots, r$, where $r = \operatorname{rank}(f(B))$. We decompose U, Σ, and V as follows:

$$U = [U_1, U_2], \qquad \Sigma = \operatorname{diag}(D, 0), \qquad V = [V_1, V_2], \tag{4}$$

where $U_1 \in R^{n\times r}$, $D \in R^{r\times r}$, $V_1 \in R^{m\times r}$, and $D = \operatorname{diag}(\sigma_{11}, \sigma_{22}, \ldots, \sigma_{rr})$, $\sigma_{11} \geq \sigma_{22} \geq \cdots \geq \sigma_{rr} > 0$. Using (1), (2), and (3), we have

$$TU_1 = (AB - I_{n\times m})V_1 D^{-1}. \tag{5}$$

We concatenate U_1 with U_2 and obtain

$$T[U_1, U_2] = \left[(AB - I_{n\times m})V_1 D^{-1}, Y\right], \tag{6}$$

where Y is an appropriate $n \times (n - r)$ matrix. So, we have

$$T = (AB - I_{n\times m})V_1 D^{-1} U_1^\top + Y U_2^\top. \tag{7}$$

Next, we shall present the condition for x, when (1) is asymptotically stable. For $x_i^s \in R^n$, we define the matrices G_i as the $n \times n$ Jacobian matrix of $f(x)$ at x_i^s

$$G_i = \frac{\partial f}{\partial x}(x_i^s), \tag{8}$$

where $i = 1, 2, \ldots, m$. We expand the right side of (1) in Taylor series about m_i, $i = 1, 2, \ldots, m$, to obtain the following.

$$\dot{x} = -Ax_i^s + Tf(x_i^s) + I + (TG_i - A)(x - x_i^s) + o\left(\|x - x_i^s\|^2\right), \tag{9}$$

where $o(\|x - x_i^s\|) \to 0$ as $x \to x_i^s$. Using $-Ax_i^s + Tf(x_i^s) + I = 0$, we obtain

$$\dot{x} = (TG_i - A)(x - x_i^s) + o\left(\|x - x_i^s\|^2\right). \tag{10}$$

Hence, from (8),(10), we conclude that if $\operatorname{Re}\lambda(TG_i - A) < 0$, then x_i^s is an asymptotically stable point for (1).

Using the estimation formula of eigenvalues in [6,7] and noticing that A is a diagonal matrix, we obtain

$$\begin{aligned}
\operatorname{Re}\lambda(TG_i - A) &\leq \lambda_{\max}\left(\frac{TG_i + (TG_i)^\top}{2}\right) + \lambda_{\max}(-A) \\
&\leq \sigma_{\max}\left(\frac{TG_i + (TG_i)^\top}{2}\right) - \sigma_{\min}(A) \\
&= \left\|\frac{TG_i + (TG_i)^\top}{2}\right\|_2 - \sigma_{\min}(A) \\
&\leq \|TG_i\|_2 - \sigma_{\min}(A).
\end{aligned} \tag{11}$$

So, if

$$\|TG_i\|_2 < \sigma_{\min}(A), \tag{12}$$

we have

$$\operatorname{Re}\lambda(TG_i - A) < 0. \tag{13}$$

Since

$$T = (AB - I_{n\times m})V_1 D^{-1} U_1^\top + Y U_2^\top. \tag{14}$$

Therefore, if

$$\|TG_i\|_2 = \left\|(AB - I_{n\times m})V_1 D^{-1} U_1^\top G_i + Y U_2^\top G_i\right\|_2 < \sigma_{\min}(A), \tag{15}$$

we have

$$\operatorname{Re}\lambda(TG_i - A) < 0. \tag{16}$$

From this, we obtain the following theorem.

THEOREM 2.1. *$x_i^s \in R^n$, $i = 1, 2, \dots, m$, are asymptotically stable equilibriums of (1), if*

$$T = (AB - I_{n\times m})V_1 D^{-1} U_1^{\top} + Y U_2^{\top}$$

and Y satisfies

$$\|Y\|_2 < \frac{\sigma_{\min}(A)}{\Lambda_{\max}} - \frac{\sigma_{\max}(AB - I_{n\times m})}{\sigma_{\min}(f(B))},$$

where $\Lambda_{\max} = \max(\sigma_{\max}(G_1), \sigma_{\max}(G_2), \dots, \sigma_{\max}(G_m))$, $B = [x_1^s, x_2^s, \dots, x_m^s]$.

PROOF. If

$$\|Y\|_2 < \frac{\sigma_{\min}(A) - \|AB - I_{n\times m}\|_2 \|D^{-1}\|_2 \|G_i\|_2}{\|G_i\|_2}. \tag{17}$$

We obtain

$$\|AB - I_{n\times m}\|_2 \left\|D^{-1}\right\|_2 \|G_i\|_2 + \|Y\|_2 \|G_i\|_2 < \sigma_{\min}(A), \tag{18}$$

note that $\|U_1\|_2 = \|U_2\|_2 = \|V_1\|_2 = 1$ and the induced 2-norm is compatible (i.e., $\|AB\|_2 \le \|A\|_2\|B\|_2$), therefore,

$$\left\|(AB - I_{n\times m})V_1 D^{-1} U_1^{\top} G_i + Y U_2^{\top} G_i\right\|_2 \le \|AB - I_{n\times m}\|_2 \left\|D^{-1}\right\|_2 \|G_i\|_2 + \|Y\|_2 \|G_i\|_2.$$

Using (14)–(16), we know that if (17) holds, then $\operatorname{Re}\lambda(TG_i - A) < 0$.

Since

$$\begin{aligned}
\frac{\sigma_{\min}(A) - \|AB - I_{n\times m}\|_2 \left\|D^{-1}\right\|_2 \|G_i\|_2}{\|G_i\|_2} &= \frac{\sigma_{\min}(A)}{\|G_i\|_2} - \frac{\|AB - I_{n\times m}\|_2}{\sigma_{\min}(D)} \\
&> \frac{\sigma_{\min}(A)}{\Lambda_{\max}} - \frac{\|AB - I_{n\times m}\|_2}{\sigma_{\min}(f(B))} \\
&= \frac{\sigma_{\min}(A)}{\Lambda_{\max}} - \frac{\sigma_{\max}(AB - I_{n\times m})}{\sigma_{\min}(f(B))}.
\end{aligned} \tag{19}$$

Therefore, if

$$\|Y\|_2 < \frac{\sigma_{\min}(A)}{\Lambda_{\max}} - \frac{\sigma_{\max}(AB - I_{n\times m})}{\sigma_{\min}(f(B))},$$

then $\operatorname{Re}\lambda(TG_i - A) < 0$, i.e., x_i^s, $i = 1, 2, \dots, m$ are asymptotically stable equilibriums of (1).

Consider

$$\dot{x} = -\left[\frac{1}{r}\right] x + Tf(x), \tag{20}$$

where

$$f_i(x_i) = \frac{1 - e^{-kx_i}}{1 + e^{-kx_i}}$$

$x_i \in R$, $k > 0$, and $r \in R^+$ is the time constant of the network, for x and $f(x)$, see (1). We have the following corollary.

COROLLARY 2.1. *(See [2].) $x_i^s \in R^n$, $i = 1, 2, \dots, m$, are asymptotically stable equilibrium of (20), if*

$$T = \left[\frac{1}{r}\right] B V_1 D^{-1} U_1^{\top} + Y U_2^{\top}$$

and Y satisfies

$$\|Y\|_2 < \frac{1}{r\Lambda_{\max}} - \frac{\sigma_{\max}(B)}{r\sigma_{\min}(f(B))},$$

where $\Lambda_{\max} = \max(\sigma_{\max}(G_1), \sigma_{\max}(G_2), \dots, \sigma_{\max}(G_m))$.

PROOF. Let $A = [1/r]E$ (E is the unit matrix), $I_{n\times m} = 0$ in Theorem 3.1. We have $\sigma_{\min}(A) = 1/r$, $\sigma_{\max}(AB - I_{n\times m}) = (1/r)\sigma_{\max}(B)$. Hence,

$$\|Y\|_2 < \frac{1}{r\Lambda_{\max}} - \frac{\sigma_{\max}(B)}{r\sigma_{\min}(f(B))}.$$

Next, we shall use generalized inverse matrix theory to present some results for system (1) in this section, related results for discrete-time neural networks has been studied by [8].

THEOREM 2.2. *$x_i^s \in R^n$, $i = 1, 2, \ldots, m$, are asymptotically stable equilibrium points of (1), if*

$$T = (AB - I_{n\times m})f^{[1]}(B) + Y(E - f(B)f^{[1]}(B))$$

and Y satisfies

$$\|Y\|_2 < \frac{\sigma_{\min}(A) - \sigma_{\max}\left[(AB - I_{n\times m})\, f^{[1]}(B)\right] \Lambda_{\max}}{\sigma_{\max}(E - f(B)f^{[1]}(B))\Lambda_{\max}},$$

where $f(B)f^{[1]}(B) \neq E$, and $\Lambda_{\max} = \max(\sigma_{\max}(G_1), \sigma_{\max}(G_2), \ldots, \sigma_{\max}(G_m))$.

PROOF. Since $Tf(B) = AB - I_{n\times m}$ Using Lemma 1.1, we obtain

$$T = (AB - I_{n\times m})f^{[1]}(B) + Y\left(E - f(B)f^{[1]}(B)\right),$$

where Y is an arbitrary matrix, and $(AB - I_{n\times m})f^{[1]}(B)f(B) = AB - I_{n\times m}$. From (12), we know that if $\|TG_i\|_2 < \sigma_{\min}(A)$, we have $\operatorname{Re}\lambda(TG_i - A) < 0$.

Since

$$\begin{aligned}\|TG_i\|_2 &= \left\|\left[(AB - I_{n\times m})\, f^{[1]}(B) + Y\left(E - f(B)f^{[1]}(B)\right)\right] G_i\right\|_2 \\ &\le \left\|(AB - I_{n\times m})f^{[1]}(B)\right\|_2 \Lambda_{\max} + \|Y\|_2 \left\|E - f(B)f^{[1]}(B)\right\|_2 \Lambda_{\max}.\end{aligned}$$

So, if

$$\left\|(AB - I_{n\times m})f^{[1]}(B)\right\|_2 \Lambda_{\max} + \|Y\|_2 \left\|E - f(B)f^{[1]}(B)\right\|_2 \Lambda_{\max} < \sigma_{\min}(A)$$

holds, that is,

$$\begin{aligned}\|Y\|_2 &< \frac{\sigma_{\min}(A) - \left\|\left((AB - I_{n\times m})f^{[1]}(B)\right)\right\|_2 \Lambda_{\max}}{\left\|\left(E - f(B)f^{[1]}(B)\right)\right\|_2 \Lambda_{\max}} \\ &= \frac{\sigma_{\min}(A) - \sigma_{\max}\left((AB - I_{n\times m})f^{[1]}(B)\right) \Lambda_{\max}}{\sigma_{\max}\left(E - f(B)f^{[1]}(B)\right) \Lambda_{\max}},\end{aligned}$$

then $\operatorname{Re}\lambda(TG_i) < 0$.

In Theorems 2.1 and 2.2, the equations guarantee the vectors to be stored are equilibrium points of the neural network. The inequalities guarantee the asymptotic stability of each equilibrium point. For a given matrix A and a given input I, it is easy for us to design a neural network.

Consider equation (20), we have the following corollary.

COROLLARY 2.2. *$x_i^s \in R^n$, $i = 1, 2, \ldots, m$, are asymptotically stable equilibrium points of (20), if*

$$T = \frac{1}{r}Bf^{[1]}(B) + Y\left(E - f(B)f^{[1]}(B)\right)$$

and Y satisfies

$$\|Y\|_2 < \frac{1 - \sigma_{\max}\left(f^{[1]}(B)\right)}{r\sigma_{\max}\left(E - f(B)f^{[1]}(B)\right)},$$

where $B = [x_1^s, x_2^s, \ldots, x_m^s]$, E is the identity matrix.

From the inequality, we conclude that the smaller the r is, the bigger the right-hand side of the inequality, and hence the larger the degree of freedom in choosing T.

If $f(B)f^{[1]}(B) = E$ or $Y = 0$, we have the following theorem.

THEOREM 2.3. *$x_i^s \in R^n, i = 1, 2, \ldots, m$. are asymptotically stable equilibrium points of (1), if*

$$T = (AB - I_{n\times m})f^{[1]}(B)$$

and

$$\sigma_{\max}\left((AB - I_{n\times m})f^{[1]}(B)\right) \Lambda_{\max} < \sigma_{\min}(A),$$

where $\Lambda_{\max} = \max(\sigma_{\max}(G_1), \sigma_{\max}(G_2), \ldots, \sigma_{\max}(G_M))$.

PROOF. Obvious.

COROLLARY 2.3. *$x_i^s \in R^n$, $i = 1, 2, \dots, m$ are equilibrium points of (20), if*

$$T = \left[\frac{1}{r}\right] B f^{[1]}(B) \quad \text{and} \quad \sigma_{\max}\left(B f^{[1]}(B)\right) \Lambda_{\max} < r\sigma_{\min}(A),$$

where $\Lambda_{\max} = \max(\sigma_{\max}(G_1), \sigma_{\max}(G_2), \dots, \sigma_{\max}(G_M))$.

EXAMPLE. Consider the model given by (1) with $f(x) = 1/(1+e^{-x})$, $I = 0$, $n = 2$, $m = 1$, and $A = E_2$ (the identity matrix). Let $x_1^s = B = (\ln 0.2\, 0)'$ be the vector to be stored, then $f(B) = ((1/6)(1/2))'$.

(1) For the SVD method, $f'(B) = \mathrm{diag}[5/36\ 0.25]$, from Theorem 2.1, the connection matrix T can be designed as

$$T = \begin{bmatrix} 0.6\ln 0.2 - \dfrac{3y_1}{\sqrt{10}} & 1.8\ln 0.2 + \dfrac{y_2}{\sqrt{10}} \\ \dfrac{-3y_1}{\sqrt{10}} & \dfrac{y_2}{\sqrt{10}} \end{bmatrix}$$

for any $Y = (y_1\, y_2)' \in R^2$ with $y_1^2 + y_2^2 \le (4 - 6\ln 5/\sqrt{10})^2 \approx 0.895495$. I choose $y_1 = y_2 = 0.6$; $y_1 = y_2 = 0$, respectively, and make a simulation, see Figure 1.

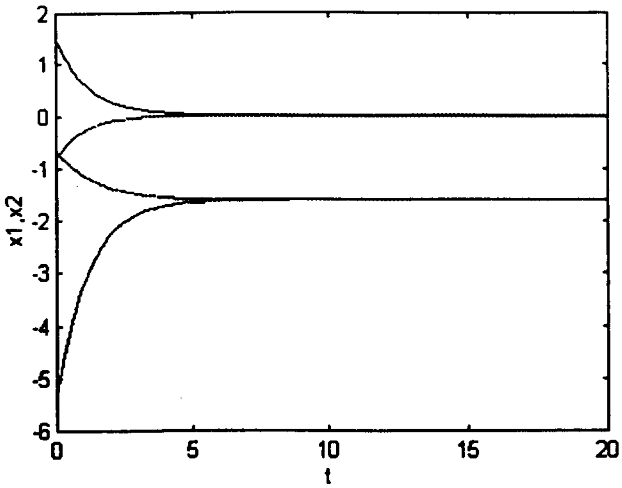

Figure 1.

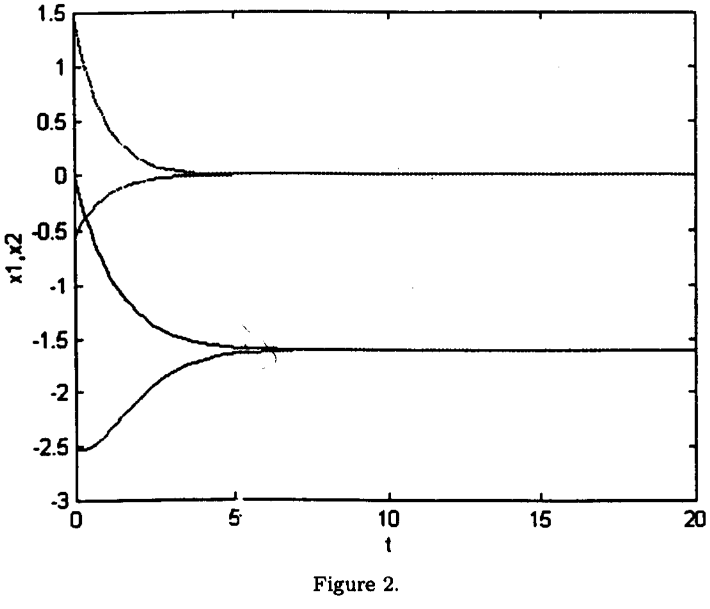

Figure 2.

(2) For the GI method, $f^{[1]}(B) = (0\,2)$, from Theorem 2.2, we know that when $y_1^2 + y_2^2 \leq (12 - 6\ln 5)^2/10 \approx 0.549139$, and

$$T = \begin{bmatrix} y_1 & 2\ln 0.2 - \dfrac{y_1}{3} \\ y_2 & \dfrac{-y_2}{3} \end{bmatrix}$$

the vector $x_1^s = (\ln 0.2\,0)'$ will be an asymptotically stable equilibrium point of (1). I choose $y_1 = y_2 = 0.1$; $y_1 = y_2 = 0.5$, respectively, and make a simulation, see Figure 2

REMARK. When $n = 2$, $m = 1$, $A = \mathrm{diag}[50\,0.1]$, B, I, and $f(x)$ defined as that in the previous example, then $\sigma_{\min}(A) = 0.1$, $\Lambda_{\max} = 0.25$, $AB - I_2 = (50\ln 0.2\,0)'$,

$$\frac{\sigma_{\max}(AB - I_{2\times 1})}{\sigma_{\min}(f(B))} = \frac{50\ln 5}{\sqrt{10}/6} \approx 152.68,$$

while $\sigma_{\min}(A)/\Lambda_{\max} = 0.1/0.25 = 0.4$. In this case, the right-hand side of the inequality in Theorem 2.1 is negative, so, the sufficient conditions given by Theorems 2.1 and 2.2 can become conservative.

3. CONCLUSION

Using the methods of SVD and GI in theory of matrices, some new techniques for designing associative memories are presented. For continuous-time neural networks, we presented some new conditions that guarantee the asymptotic stability of equilibrium points. Using the results in this paper, we can design associative memories. In the previous papers, most of the methods produce symmetric interconnection matrices. In hardware implementations, the symmetric interconnection matrices will not be realized perfectly. The methods in this paper do not rely on symmetry. Therefore, the neural networks designed by the methods are easier to implement in hardware. In the future, we would like to study the optimal design of neural networks for associative memories with each of these two methods.

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